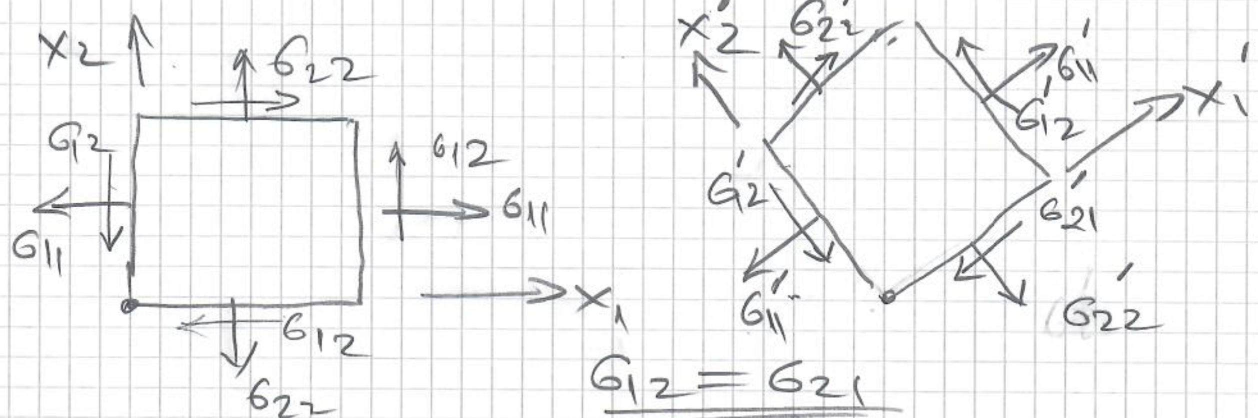
Origin of Mohr's circle

Stress-state at a point of a solid with respect to two coordinate systems: Ox_1x_2, $Ox_1'x_2'$



$$\underline{\sigma_{12} = \sigma_{21}}$$

The following relations between the stresses are true:

(a) $\sigma_{11}' = \sigma_{11}\cos^2\vartheta + \sigma_{22}\sin^2\vartheta + 2\sigma_{12}\cos\vartheta\sin\vartheta$

(b) $\sigma_{22}' = \sigma_{11}\sin^2\vartheta + \sigma_{22}\cos^2\vartheta - 2\sigma_{12}\cos\vartheta\sin\vartheta$

(c) $\sigma_{12}' = (\sigma_{22} - \sigma_{11})\cos\vartheta\sin\vartheta + \sigma_{12}(\cos^2\vartheta - \sin^2\vartheta)$

We use the following trigonometric identities to transform (a),(c).

$$\cos^2\vartheta = \frac{1+\cos 2\vartheta}{2};$$

$$\sin^2\vartheta = \frac{1-\sin 2\vartheta}{2}$$

$$2\cos\vartheta\sin\vartheta = \sin 2\vartheta$$

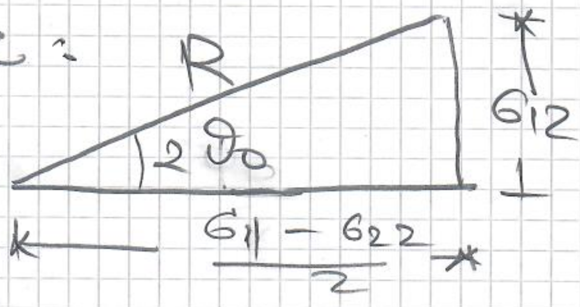
$\Longrightarrow$

$$\sigma_{11}' = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\vartheta + 2\sigma_{12}\sin 2\vartheta$$

$$\sigma_{12}' = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\vartheta + \sigma_{12}\cos 2\vartheta$$

Define $R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$

and a triangle:



$$\Longrightarrow \quad \frac{(\sigma_{11}-\sigma_{22})/2}{R} = \cos 2\vartheta_0$$

$$\sigma_{12}/R = \sin 2\vartheta_0$$

with these parameters σ_{11}', σ_{12}' can be expressed as:

$$\sigma_{11}' - \frac{\sigma_{11}+\sigma_{22}}{2} = R\left[\frac{(\sigma_{11}-\sigma_{22})/2}{R}\cos 2\vartheta + \frac{\sigma_{12}}{R}\sin 2\vartheta\right]$$

$$\sigma_{12}' = R\left[-\frac{(\sigma_{11}-\sigma_{22})/2}{R}\sin 2\vartheta + \frac{\sigma_{12}}{R}\cos 2\vartheta\right]$$

or

$$\sigma_{11}' - \frac{\sigma_{11}+\sigma_{22}}{2} = R\left[\cos 2\vartheta_0 \cos 2\vartheta + \sin 2\vartheta_0 \sin 2\vartheta\right]$$

$$\sigma_{12}' = R\left[-\cos 2\vartheta_0 \sin 2\vartheta + \sin 2\vartheta_0 \cos 2\vartheta\right]$$

[Recall: $\cos(a-b) = \cos a \cos b + \sin a \sin b$
$\sin(a-b) = \sin a \cos b - \sin b \cos a$]

$$\sigma_{11}' - \frac{\sigma_{11} + \sigma_{22}}{2} = R \cdot \cos(2\theta - 2\theta_0)$$

$$\sigma_{12}' = R\left[\sin(2\theta - 2\theta_0)\right]$$

take the squares of both sides in each expression and add.

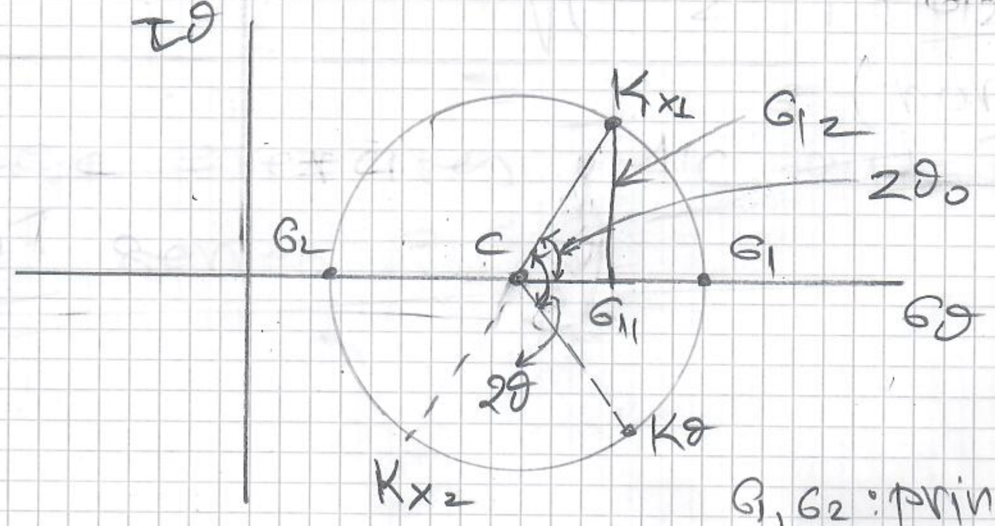
$$\left(\sigma_{11}' - \frac{\sigma_{11} + \sigma_{22}}{2}\right)^2 + \sigma_{12}'^2 = R^2$$

This last equation is that of a circle on σ_{11}', σ_{12}' plane with center $C\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right)$

Radius $R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$

For simplicity

$\sigma_{11}' \longrightarrow \sigma\theta$, $\sigma_{12}' \longrightarrow \tau\theta$



σ_1, σ_2: principal stresses

θ_0: orientation of σ_1